Fuzzy Difference-in-Differences (de Chaisemartin & D'Haultfœuille, REStud 2018)

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Introduction

- Goal: study **instrumented difference-in-differences** with **heterogeneity in treatment effects**
- **Fuzzy DiD** is a commonly used design
  - But it is not always explicitly recognized as such in empirical work (e.g. Duflo 2001)
- In a related article, Blundell and Costa Dias (2009) discusses fuzzy DiD with heterogeneity

Today's Presentation

1. Identification of a (conditional) LATE with a **standard** Wald estimand and **strong** assumptions

2. Identification of a (conditional) LATE with a **corrected** Wald estimand and **other** assumptions

3. Extension to designs with **multiple treatment groups**

1. Model Setup

2. Identification Results

3. Extension to Multiple Treatment Groups

Model Setup

- $T \in \{0, 1\}$ denotes **time**, $G \in \{0, 1\}$ indicates a **time-invariant group**
- D is a **binary treatment**, **not** a **deterministic** function of G and T, i.e., $D \neq GT$
- $Y \in \mathbb{R}$ is an **outcome** of interest
- $Z \equiv GT$ is a **binary instrument** (not nested in Hudson, Hull, and Liebersohn 2017)
- Unlike sharp designs, **fuzzy** designs allow for
  - Units to be treated in the control group ($G = 0$)
  - Units to be treated (in either group) in $T = 0$

The IV-DiD Wald Estimand

- **Reduced-form** and **first-stage** (saturated) linear regressions:

$$Y = \alpha + \beta G + \gamma T + \delta GT + U$$
$$D = \lambda + \eta G + \phi T + \rho GT + V$$

- The coefficient associated with D in the structural equation is the **IV-DiD Wald estimand**:

$$\omega \equiv \frac{\Delta_Y(1) - \Delta_Y(0)}{\Delta_D(1) - \Delta_D(0)}$$

where $\Delta_Y(g)$ and $\Delta_D(g)$ denote **time trends** in group $g \in \{0, 1\}$

Further Assumptions

Without loss of generality, define D and G such that

1. The treatment rate **increases** over time in the **treated** group,

$$\mathbb{E}[D|G=1, T=1] - \mathbb{E}[D|G=1, T=0] > 0$$

2. The treatment rate in the **control** group **does not increase more** than in the **treated** group,

$$\mathbb{E}[D|G=1, T=1] - \mathbb{E}[D|G=1, T=0] > \mathbb{E}[D|G=0, T=1] - \mathbb{E}[D|G=0, T=0]$$

Potential Outcomes and Potential Treatments

- D and Y are linked by **potential outcomes** $Y(0), Y(1)$
- **Potential treatments** are $D(0), D(1)$, where $D = D(t)$ is observed (G is subsumed)
  - Both potential treatments are **observed** in a repeated cross section
  - Potential treatments are **independent of time within each group**
- Within each group, units **switch** treatment only in **one direction**:

$$\mathbb{P}(D(1) \geq D(0)|G) = 1 \quad \text{or} \quad \mathbb{P}(D(1) \leq D(0)|G) = 1$$

Identification Result #1

- Further assume that
  - **Common trends** holds, i.e., $\mathbb{E}[Y(0)|G,T=1]-\mathbb{E}[Y(0)|G,T=0]$ does not depend on G
  - The **ATE** among units **treated in the pre-period** is **stable** over time **within** each group, i.e.,

$$\mathbb{E}[Y(1)-Y(0)|G,T=1,D(0)=1]=\mathbb{E}[Y(1)-Y(0)|G,T=0,D(0)=1]$$

- Then the IV-DiD Wald estimand identifies **a weighted average of two causal parameters**:
  1. $\tau_1 \equiv \mathbb{E}[Y(1)-Y(0)|G=1,T=1,D(1)>D(0)]$, the ATE among **treated "switchers"**
  2. $\tau_0 \equiv \mathbb{E}[Y(1)-Y(0)|G=0,T=1,D(1)\neq D(0)]$, the ATE among **control "switchers"**

Identification Result #1

- Case (a): the treatment rate **increases** in the control group. Then

$$\omega = \alpha\tau_1 + (1-\alpha)\,\tau_0 \quad \text{with} \quad \alpha \equiv \frac{\mathbb{P}\left(D(1) > D(0)|G=1\right)}{\mathbb{P}\left(D(1) > D(0)|G=1\right) - \mathbb{P}\left(D(1) > D(0)|G=0\right)}$$

$\alpha > 1 \implies$ the IV-DiD estimand **negatively weights** the ATE among **control switchers**

- Case (b): the treatment rate **decreases** in the control group. Then

$$\omega = \alpha\tau_1 + (1-\alpha)\,\tau_0 \quad \text{with} \quad \alpha \equiv \frac{\mathbb{P}\left(D(1) > D(0)|G=1\right)}{\mathbb{P}\left(D(1) > D(0)|G=1\right) + \mathbb{P}\left(D(1) < D(0)|G=0\right)}$$

$\alpha \in (0,1) \implies$ the IV-DiD estimand is a **convex combination** of **ATEs among switchers**

Identification Result #2

- **Modify** the assumptions made for identification result #1

- Instead, assume some version of **conditional common trends**:

$$\mathbb{E}\left[Y(d)|G, T=1, D(0)=d\right] - \mathbb{E}\left[Y(d)|G, T=0, D(0)=d\right]$$

  does not depend on G for $d \in \{0,1\}$

- Consider a **"time-corrected" Wald estimand**:

$$\omega_{\text{TC}} \equiv \frac{\mathbb{E}\left[Y|G=1, T=1\right] - \mathbb{E}\left[Y + (1-D)\delta_0 + D\delta_1 | G=1, T=0\right]}{\mathbb{E}\left[D|G=1, T=1\right] - \mathbb{E}\left[D|G=1, T=0\right]}$$

  with

$$\delta_d \equiv \mathbb{E}\left[Y|D=d, G=0, T=1\right] - \mathbb{E}\left[Y|D=d, G=0, T=0\right] \quad \text{for } d \in \{0,1\}$$

Identification Result #2

- ω_{TC} identifies τ_1, the ATE among **switchers in the treated group**
- Intuition behind this "correction":
  - Begin with $\mathbb{E}\left[Y|G=1, T=0\right]$
  - Add $\delta_0 \times \mathbb{P}\left(D=0|G=1, T=0\right)$ and $\delta_1 \times \mathbb{P}\left(D=1|G=1, T=0\right)$
  - Obtain a counterfactual $\mathbb{E}\left[Y|G=1, T=1\right]$ purged from the contribution of switchers
  - Contrast between $\mathbb{E}\left[Y|G=1, T=1\right]$ and "corrected" $\mathbb{E}\left[Y|G=1, T=0\right] \rightarrow$ switchers
  - Scale the numerator by the evolution of the treatment rate in the treated group

Extension to Multiple Treatment Groups

These identification results can be extended to situations with multiple treatment groups

- **Groups** $G \in \{0, 1, \dots, \overline{g}\}$

- Partition them into **three super-groups** based on how the treatment rate evolves
  - Group g belongs to $\mathcal{G}_i$, $\mathcal{G}_s$, or $\mathcal{G}_d$ if $\Delta_D(g)$ increases, is stable, or decreases

- The target parameter becomes the **ATE among all switchers**, i.e.,

$$\tau^* \equiv \mathbb{E}\left[Y(1) - Y(0) \middle| T = 1, \bigcup_{g=0}^{\overline{g}} \{D(0) \neq D(1), G = g\} \right]$$

- For compactness of notation, define $G^* \equiv \mathbb{I}[G \in \mathcal{G}_i] - \mathbb{I}[G \in \mathcal{G}_d] \in \{-1, 0, 1\}$

Extension to Multiple Treatment Groups

Under the same assumptions as in **identification result #1**, τ^* is identified as follows:

1. Compute **four difference-in-differences contrasts**:

$$\text{DiD}_R^* (g, g') \equiv \Delta_R (g) - \Delta_R (g')$$

where R is either Y or D and $(g, g') \in \{(1,0), (0,-1)\}$

2. Compute two Wald estimands by taking **ratios of DiD contrasts**:

$$\omega^*_{\text{DiD}}(1,0) \equiv \frac{\text{DiD}_Y^* (1,0)}{\text{DiD}_D^* (1,0)} \qquad \omega^*_{\text{DiD}}(0,-1) \equiv \frac{\text{DiD}_Y^* (0,-1)}{\text{DiD}_D^* (0,-1)}$$

3. Compute a **convex combination** of these two Wald estimands:

$$\omega^*_{\text{DiD}} \equiv \theta \omega^*_{\text{DiD}}(1,0) + (1-\theta)\, \omega^*_{\text{DiD}}(0,-1)$$

Extension to Multiple Treatment Groups

Under the same assumption as in **identification result #2**, τ^* is identified as follows:

1. Compute **two time correction terms** (for $d \in \{0, 1\}$):

$$\delta_d^* \equiv \mathbb{E}\left[Y|D = d, G^* = 0, T = 1\right] - \mathbb{E}\left[Y|D = d, G^* = 0, T = 0\right]$$

2. Compute **two time-corrected Wald ratios** (for $g \in \{-1, 1\}$):

$$\omega_{\text{TC}}^*(g) \equiv \frac{\mathbb{E}\left[Y|G^* = g, T = 1\right] - \mathbb{E}\left[Y + (1 - D)\delta_0^* + D\delta_1^*|G^* = g, T = 0\right]}{\mathbb{E}\left[D|G^* = g, T = 1\right] - \mathbb{E}\left[D|G^* = g, T = 0\right]}$$

3. Compute a **convex combination** of these two time-corrected Wald ratios:

$$\omega_{\text{TC}}^* \equiv \theta\omega_{\text{TC}}^*(1) + (1 - \theta)\,\omega_{\text{TC}}^*(-1)$$

Extension to Multiple Treatment Groups

- In practice, super-groups $G^* \in \{-1, 0, 1\}$ may be **known** ex ante or need to be **estimated**
  - Estimation is likely necessary when the treatment varies at the unit level
- The authors develop a **data-based procedure** to classify groups into three super-groups
  - This hinges on running t-tests within each group to compare the treatment rate over time
- Once super-groups have been determined, target parameters can be estimated