Estimating dynamic treatment effects in event studies with heterogeneous treatment effects

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Static TWFE

The static regression we saw last week

$$Y_{it} = \alpha_i + \gamma_t + \beta D_{it} + \varepsilon_{it} \tag{1}$$

for $i = 1, \cdots, N$ and $t = 1, \cdots, T$.

Let us denote the treatment timing of unit i with E_i: $D_{it} = \mathbf{1}_{\{t \geq E_i\}}$.

Dynamic TWFE

We can make a dynamic regression by having

$$Y_{it} = \alpha_i + \gamma_t + \sum_{r=1}^{R} \beta_r \mathbf{1}_{\{t-E_i \in G_r\}} + \varepsilon_{it}, \tag{2}$$

where $G_1, \cdots, G_R$ are disjoint subsets of $\{-T, \cdots, T\}$. $t - E_i$ is the *relative* treatment timing.

For the regression to have no multicollinearity,

- $\sum_{i,t} \mathbf{1}_{\{t-E_i \in G_r\}} > 0$ for all r and
- $\sum_r \sum_{i,t} \mathbf{1}_{\{t-E_i \in G_r\}} < NT$.

An example is

$$\{-4\}, \{-3\}, \{-2\}, \{0\}, \{1\}, \{2\}, \{3\}, \{4, 5, \cdots\}.$$

We have three pretreatment periods and five treatment periods. Some treatment periods are pooled.

FWL theorem

Sun and Abraham (JoE, 2021) uses the FWL theorem as in Goodman-Bacon (JoE, 2021).

However, the integrands are different.

Let us first discuss the static case. From the FWL theorem, regressing

$$Y_{it} = \beta \tilde{D}_{it} + \varepsilon_{it} \tag{3}$$

where $\tilde{D}_{it}$ is the residual from projecting out α_i and γ_t gives us the same $\hat{\beta}$.

$$\tilde{D}_{it} = D_{it} - \frac{1}{N}\sum_{j=1}^{N} D_{jt} - \frac{1}{T}\sum_{s=1}^{T} D_{is} + \frac{1}{NT}\sum_{j,s} D_{js}.$$

With matrix notation and $\mathbb{M}_{FE}$ being the residual generator from FE projection,

$$\hat{\beta} = \left(\mathbb{D}^{\intercal}\mathbb{M}_{FE}\mathbb{D}\right)^{-1}\mathbb{D}^{\intercal}\mathbb{M}_{FE}\mathbb{Y}$$

FWL theorem

Note that

$$\mathbb{M}_{FE}\mathbb{Z}^1 = \mathbb{M}_{FE}\left(Z_{it}^1\right)_{i,t} = \mathbb{M}_{FE}\left(Y_{i0}\right)_{i,t} = \mathbf{0},$$

$$\mathbb{M}_{FE}\mathbb{Z}^2 = \mathbb{M}_{FE}\left(Z_{it}^2\right)_{i,t} = \mathbb{M}_{FE}\left(\mathbf{E}\left[Y_{it} - Y_{i0}|E_i = \infty\right]\right)_{i,t} = \mathbf{0},$$

since $Z_{it}^1 = Y_{i0}$ is only a function of i and $Z_{it}^2 = \mathbf{E}\left[Y_{it} - Y_{i0}|E_i = \infty\right]$ is only a function of t.

Thus,

$$\hat{\beta} = \left(\mathbb{D}^{\intercal}\mathbb{M}_{FE}\mathbb{D}\right)^{-1}\mathbb{D}^{\intercal}\mathbb{M}_{FE}\left(\mathbb{Y} - \mathbb{Z}^1 - \mathbb{Z}^2\right)$$

and regressing (3) is equal to regressing

$$Y_{it} - Y_{i0} - \mathbf{E}\left[Y_{it} - Y_{i0}|E_i = \infty\right] = \beta\tilde{D}_{it} + \varepsilon_{it}. \tag{4}$$

FWL theorem

Lastly, note that $\tilde{D}_{it} = \tilde{D}_{jt}$ whenever $E_i = E_j$.

We can group units based on their treatment timing. Then,

$$\hat{\beta} = \frac{1}{\sum_{i,t} \tilde{D}_{it}^2} \sum_{i,t} \tilde{D}_i \cdot (Y_{it} - Y_{i0} - \mathbf{E}[Y_{it} - Y_{i0} | E_i = \infty])$$
$$= \sum_{e,t} s_t^e \cdot \left(\frac{1}{N_e} \sum_{i=1}^{N} (Y_{it} - Y_{i0} - \mathbf{E}[Y_{it} - Y_{i0} | E_i = \infty]) \mathbf{1}_{\{E_i = e\}} \right)$$

where $N_e = \sum_{i=1}^{N} \mathbf{1}_{\{E_i = e\}}$ and under some regularity conditions,

$$\hat{\beta} \xrightarrow{p} \beta = \sum_{e,t} \sigma_t^e \cdot (\mathbf{E}[Y_{it} - Y_{i0} | E_i = e] - \mathbf{E}[Y_{it} - Y_{i0} | E_i = \infty]) =: \sum_{e,t} \sigma_t^e \cdot DID_t^e.$$

where σ_t^e is the population version of s_t^e.

Negative weighting problem

Note that the integrand DID_t^e is different from Goodman-Bacon (JoE, 2021).

I cannot say ATT_t^e since for $t < e$, DID_t^e is not treatment effect...

For $t \geq e$, under the parallel trend, $DID_t^e = \mathbf{E}\left[Y_{it}(1) - Y_{it}(0) | E_i = e\right]$.

This idea of applying the FWL theorem to TWFE is *not* unique to this paper.

de Chaisemartin and D'Haultfœuille (AER, 2020), Goodman-Bacon (JoE, 2021),

Borusyak, Jaravel, Spiess (wp, 2022) all derived similar decomposition and pointed out this problem.

Sun and Abraham (JoE, 2021) built on this and discussed dynamic case explicitly.

Negative weighting problem

Note that the empirical weights s_t^e is

$$\left(\frac{1}{N_{E_i}} s_t^{E_i}\right)_{i,t} = \left(\mathbb{D}^\intercal \mathbb{M}_{FE} \mathbb{D}\right)^{-1} \mathbb{D}^\intercal \mathbb{M}_{FE}.$$

We have some good properties on s_t^e.

1) Weights on treatment periods sum to 1.

$$\sum_{e,t} s_t^e \cdot \mathbf{1}_{\{t \geq e\}} = \sum_{i,t} \frac{1}{N_{E\,i}} s_t^{E_i} \cdot D_{it} = \left(\mathbb{D}^\intercal \mathbb{M}_{FE} \mathbb{D}\right)^{-1} \mathbb{D}^\intercal \mathbb{M}_{FE} \mathbb{D} = 1.$$

2) Weigts on pretreatment periods sum to -1.

$$\sum_{e,t} s_t^e = \sum_{i,t} \frac{1}{N_{E\,i}} s_t^{E_i} = \left(\mathbb{D}^\intercal \mathbb{M}_{FE} \mathbb{D}\right)^{-1} \mathbb{D}^\intercal \mathbb{M}_{FE} \mathbf{1} = 0 \quad \text{and thus} \quad \sum_{e,t} s_t^e \cdot \mathbf{1}_{\{t < e\}} = -1.$$

However, s_t^e has a *negative weighting* problem.

Negative weighting problem

Suppose $T = 5$ and $E_i \in \{2, 3, 4, 5\}$.

	Jake	Amy	Raymond	Cheddar
1	0	0	0	0
2	0	0	0	1
3	0	0	1	1
4	0	1	1	1
5	1	1	1	1

Figure 1: D_{it}

Negative weighting problem

Suppose $T = 5$ and $E_i \in \{2, 3, 4, 5\}$.

	Jake	Amy	Raymond	Cheddar
1	$-\frac{1}{5}$	$-\frac{2}{5}$	$-\frac{3}{5}$	$-\frac{4}{5}$
2	$-\frac{1}{5}$	$-\frac{2}{5}$	$-\frac{3}{5}$	$\frac{1}{5}$
3	$-\frac{1}{5}$	$-\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{5}$
4	$-\frac{1}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{5}$
5	$\frac{4}{5}$	$\frac{3}{5}$	$\frac{2}{5}$	$\frac{1}{5}$

Figure 2: $D_{it} - \frac{1}{T} \sum_{s=1}^{T} D_{is}$

Negative weighting problem

Suppose $T = 5$ and $E_i \in \{2, 3, 4, 5\}$.

	Jake	Amy	Raymond	Cheddar
1	$\frac{6}{20}$	$\frac{2}{20}$	$-\frac{2}{20}$	$-\frac{6}{20}$
2	$\frac{1}{20}$	$-\frac{3}{20}$	$-\frac{7}{20}$	$\frac{9}{20}$
3	$-\frac{4}{20}$	$-\frac{8}{20}$	$\frac{8}{20}$	$\frac{4}{20}$
4	$-\frac{9}{20}$	$\frac{7}{20}$	$\frac{3}{20}$	$-\frac{1}{20}$
5	$\frac{6}{20}$	$\frac{2}{20}$	$-\frac{2}{20}$	$-\frac{6}{20}$

Figure 3: $\tilde{D}_{it} \propto s_t^{E_i}$ for each i

Always, the empirical weight s_t^e is decreasing in t, with a jump at $t = e$.

It decreases in t since $\sum_i \left(D_{it} - \frac{1}{T} \sum_{s=1}^{T} D_{is} \right)$ increases in t.

We can show that $s_t^e > 0$ for $t = e$, except for some crazy cases.

Negative weighting problem

Positive weights for pretreatment periods are not that sad: we assume

$$DID_t^e = \mathbf{E}\left[Y_{it} - Y_{i0} | E_i = e\right] - \mathbf{E}\left[Y_{it} - Y_{i0} | E_i = \infty\right] = 0$$

for $t < e$ anyways. (*no anticipation*)

However, negative weights for treatment periods IS sad:

similar to CONTROL v. GOOD. v. BAD from last week.

Goodman-Bacon (JoE, 2021) summed s_t^e so that the weights are all positive.

If we expand further, the weights are not only arbitrary, but even contrary to the common sense.

The negative weighting happens for later periods of earlier-treated units:

if the treatment effect is larger for later periods, $\beta < 0$ even is possible when $DID_t^e > 0$ for all $t \geq e$.

FWL theorem: dynamic case

Recall

$$Y_{it} = \alpha_i + \gamma_t + \sum_{r=1}^{R} \beta_r \mathbf{1}_{\{t-E_i \in G_r\}} + \varepsilon_{it}.$$

Proposition 1 (Sun and Abraham)

$$\beta_r = \sum_{t-e \in G_r} \sigma_t^e(r) DID_t^e + \sum_{r' \neq r} \sum_{t-e \in G_{r'}} \sigma_t^e(r') DID_t^e + \sum_{t-e \notin \cup_{r'} G_{r'}} \sigma_t^e(r) DID_t^e. \quad (5)$$

and

$$\sum_{t-e \in G_r} \sigma_t^e(r) = 1, \quad \sum_{r' \neq r} \sum_{t-e \in G_{r'}} \sigma_t^e(r') = 0, \quad \sum_{t-e \notin \cup_{r'} G_{r'}} \sigma_t^e(r) = -1.$$

The signs of each $\sigma_t^e(r)$ are not fixed. The negative weighting problem still exists.

FWL theorem: dynamic case

Prosition 2,3,4 are all simplification of (5) with parallel trend, no anticipation, etc.

e.g., time-invariance ($DID_t^{t+l} = DID_s^{s+l}$ for all t, s, l) allows us to ignore the negative weighting.

Let us go back to the example of

$$\{G_r\}_{r=1}^{8} = \{\{-4\}, \{-3\}, \{-2\}, \{0\}, \{1\}, \{2\}, \{3\}, \{4, 5, \cdots\}\}.$$

Then, β_4, which is the treatment effect on $t = E_i$, has the following interpretation:

$$\beta_4 = A - B + C$$

A : some average of DID_t^e where $t = e$

B : some average of DID_t^e where $t - e = -1$ or $t - e < -4$

C : interference from DID_t^e where $t - e \in \{-4, -3, -2, 1, 2, \cdots\}$.

Solution

Sun and Abraham (JoE, 2021) suggests that we directly impose weights on DID_t^e:

$$\tilde{\beta}_r = \frac{1}{|G_r|} \sum_{l \in G_r} \left(\sum_{1 \le e+l \le T} \frac{N_e}{\sum_{1 \le e'+l \le T} N_{e'}} \widehat{DID}_t^e \right)$$

The inner sum is the (weighted) average of $\widehat{DID}_t^e$ for treatment timing e

such that $e - l$ is observed in the data: $1 \le e + l \le T$.

The outer sum is the average of DID estimand for relative timing $l \in G_r$.

This follows the same spirit with de Chaisemartin and D'Haultfœuille (AER, 2020).

A conditional on observable version will be discussed in the next week.

Solution

DID_t^e can either be estimated directly: with some $e' > t$

$$\widehat{DID}_t^e = \frac{1}{N_e}\sum_{i=1}^{N}(Y_{it} - Y_{i1})\,\mathbf{1}_{\{E_i=e\}} - \frac{1}{N_{e'}}\sum_{i=1}^{N}(Y_{it} - Y_{i1})\,\mathbf{1}_{\{E_i=e'\}}.$$

Without such e', DID_t^e is not estimated. Also for $t = 1$, DID_t^e is not estimated.

An indirect estimation is to use TWFE regression with treatment timing specific β:

$$Y_{it} = \alpha_i + \gamma_t + \sum_{s,e}\beta_{s,e}\mathbf{1}_{\{t=s,E_i=e\}} + \varepsilon_{it}.$$

Of course, we have to drop some s and e.

Solution

Table 3
Estimates for the effect of hospitalization on outcomes.

(a) Out-of-pocket medical spending

ℓ Wave relative to hospitalization	FE $\widehat{\mu}_\ell$	IW $\widehat{\nu}_\ell$	$\widehat{\delta}_{1,\ell}$	$CATT_{e,\ell}$ $\widehat{\delta}_{2,\ell}$	$\widehat{\delta}_{3,\ell}$
−3	149	591	–	–	591
	(792)	(1273)			(1273)
−2	203	353	–	299	411
	(480)	(698)		(967)	(1030)
−1	0	0	0	0	0
0	3013	2960	2826	3031	3092
	(511)	(543)	(1038)	(704)	(998)
1	888	530	825	107	–
	(664)	(587)	(912)	(653)	
2	1172	800	800	–	–
	(983)	(1010)	(1010)		
3	1914	–	–	–	–
	(1426)				

Figure 4: Comparison between $\hat{\beta}$, $\tilde{\beta}$ and indirect $\widehat{DID}_t^e$

My two cents

To econometricians:

- The literature is very, *very* saturated: the competition...
- That being said, being able to skip the [motivation] slide is a big appeal...
- Some variants of DID are yet to be understood: e.g. Fadlon and Nielsen (AER, 2019)
- Proposing sensible alternatives to the parallel trend is always good.

To applied microeconomists:

- Without strong assumptions, reduced form parameters are not supposed to make sense.

  TWFE won't probably die anytime soon.
- Existence of X_{it} messes up solutions suggested here.
- shameless self-promotion: wanna explore heterogeneity with long panel? Shin (2022)...